

# SMALL COVER AND HALPERIN-CARLSSON CONJECTURE

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**ABSTRACT.** We prove that the Halperin-Carlsson conjecture holds for any free  $(\mathbb{Z}_2)^m$ -action on a compact manifold whose orbit space is a small cover. In addition, we show that if the total space of a principal  $(\mathbb{Z}_2)^m$ -bundle over a small cover is connected, it must be equivalent to a partial quotient of the corresponding real moment-angle manifold with some canonical  $\mathbb{Z}_2$ -torus action.

## 1. INTRODUCTION

For any prime  $p$ , let  $\mathbb{Z}_p$  denote the quotient group  $\mathbb{Z}/p\mathbb{Z}$ . And let  $S^1$  be the circle group.

**Halperin-Carlsson Conjecture:** If  $G = (\mathbb{Z}_p)^m$  ( $p$  is a prime) or  $(S^1)^m$  acts freely on a finite dimensional CW-complex  $X$ , then  $\sum_{i=0}^{\infty} \dim_{\mathbb{Z}_p} H^i(X, \mathbb{Z}_p) \geq 2^m$  or  $\sum_{i=0}^{\infty} \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) \geq 2^m$  respectively.

The above conjecture was proposed in the middle of 1980s by S. Halperin in [1] for the torus case, and by G. Carlsson in [2] for the  $\mathbb{Z}_p$ -torus case. It is also called *toral rank conjecture* in some papers.

In the earlier time, this conjecture mainly took the form of whether a free  $(\mathbb{Z}_p)^m$ -action on a product of spheres  $S^{n_1} \times \cdots \times S^{n_k}$  implies  $m \leq k$ . Many authors have studied this intriguing conjecture and contributed results with respect to different aspects (see [3]–[8]). The reader is referred to see a survey of such results in [9] and [10]. But the general case is still open for any prime  $p$ .

For general finite dimensional CW-complexes, the conjecture has been proved in [11] for  $m \leq 3$  in the torus and  $\mathbb{Z}_2$ -torus cases and  $m \leq 2$  in the odd  $\mathbb{Z}_p$ -torus case. More recently, Cao and Lü (see [12]) and Ustinovsky (see [13]) independently proved the following result, which confirmed the Halperin-Carlsson conjecture for some canonical  $\mathbb{Z}_2$ -torus actions on real moment-angle complexes.

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**Theorem 1.1** (see [12] and [13]). *If  $K^{n-1}$  is an  $(n-1)$ -dimensional simplicial complex on the vertex set  $[d]$ . Then the real moment-angle complex  $\mathcal{Z}_{K^{n-1}}$  over  $K^{n-1}$  must satisfy:  $\sum_i \dim_{\mathbb{Z}_2} H^i(\mathcal{Z}_{K^{n-1}}, \mathbb{Z}_2) \geq 2^{d-n}$ . In particular, if  $P^n$  is an  $n$ -dimensional simple polytope with  $d$  facets, then the real moment-angle manifold  $\mathcal{Z}_{P^n}$  must satisfy:  $\sum_i \dim_{\mathbb{Z}_2} H^i(\mathcal{Z}_{P^n}, \mathbb{Z}_2) \geq 2^{d-n}$ .*

**Remark 1.2.** Indeed, much stronger results were obtained in [12] and [13]. For example, it was shown in [12] and [13] that the Theorem 1.1 holds even if the  $\mathbb{Z}_2$ -coefficient is replaced by the rational coefficient.

**Remark 1.3.** There is a purely algebraic analogue of the Halperin-Carlsson conjecture, which is proposed in [2] in the context of commutative algebras. Some related results were obtained in [14].

In this paper, we will only study the conjecture for  $G = (\mathbb{Z}_2)^m$  and  $X$  being a closed manifold. In addition, we will use the following conventions:

- (1) we always treat  $(\mathbb{Z}_2)^m$  as an additive group;
- (2) any manifold and submanifold in this paper are smooth;
- (3) we do not distinguish an embedded submanifold and its image.

Suppose  $(\mathbb{Z}_2)^m$  acts freely and smoothly on a closed  $n$ -manifold  $M^n$ . Let  $Q^n = M^n / (\mathbb{Z}_2)^m$  be the orbit space. Then  $Q^n$  is a closed  $n$ -manifold too. Let  $\pi : M^n \rightarrow Q^n$  be the orbit map. We can think of  $M^n$  either as a principal  $(\mathbb{Z}_2)^m$ -bundle over  $Q^n$  or as a regular covering over  $Q^n$  with deck transformation group  $(\mathbb{Z}_2)^m$ . In algebraic topology, we have a standard way to recover  $M^n$  from  $Q^n$  using the universal covering space of  $Q^n$  and the monodromy of the covering (see [15]). However, it is not very easy for us to visualize the total space of the covering in this approach. In [16], a new way of constructing principal  $(\mathbb{Z}_2)^m$ -bundles over closed manifolds is introduced, which allows us to visualize this kind of objects more easily.

Indeed, it is shown in [16] that  $\pi : M^n \rightarrow Q^n$  determines a  $(\mathbb{Z}_2)^m$ -coloring  $\lambda_\pi$  on a nice manifold with corners  $V^n$  (called a  $\mathbb{Z}_2$ -core of  $Q^n$ ), and up to equivariant homeomorphism, we can recover  $M^n$  by a standard *glue-back construction* from  $V^n$  and  $\lambda_\pi$ . Using this new language, we will prove the following theorem which confirms the Halperin-Carlsson conjecture in some new cases.

**Theorem 1.4.** *Suppose  $(\mathbb{Z}_2)^m$  acts freely on a closed  $n$ -manifold  $M^n$  whose orbit space is homeomorphic to a small cover, then we must have:*

$$\sum_i \dim_{\mathbb{Z}_2} H^i(M^n, \mathbb{Z}_2) \geq 2^m \quad (1)$$

Recall that an  $n$ -dimensional small cover is a closed  $n$ -manifold with a locally standard  $(\mathbb{Z}_2)^n$ -action whose orbit space is a simple polytope (see [17]).

Suppose  $P^n$  is a simple polytope with  $d$  facets. There is a *canonical action* of  $(\mathbb{Z}_2)^d$  on the real moment-angle complex  $\mathcal{Z}_{P^n}$  whose orbit space is  $P^n$ . For a subtorus  $H \subset (\mathbb{Z}_2)^d$ , if  $H$  acts freely on  $\mathcal{Z}_{P^n}$  through the canonical action,  $\mathcal{Z}_{P^n}/H$  is called a *partial quotient* of  $\mathcal{Z}_{P^n}$  (see [18]). In addition, if there is a subgroup  $\tilde{H}$  of  $(\mathbb{Z}_2)^d$  with  $\tilde{H} \supset H$  and  $\tilde{H}$  also acts freely on  $\mathcal{Z}_{P^n}$  through the canonical action, we will get an induced free action of  $\tilde{H}/H$  on  $\mathcal{Z}_{P^n}/H$  whose orbit space is  $\mathcal{Z}_{P^n}/\tilde{H}$ . By abusing of terminology, we also call this kind of  $\tilde{H}/H$ -action on  $\mathcal{Z}_{P^n}/H$  *canonical*.

In addition, two principal  $(\mathbb{Z}_2)^m$ -bundles  $M_1^n$  and  $M_2^n$  over a space  $Q^n$  are called *equivalent* if there is a homeomorphism  $f : M_1^n \rightarrow M_2^n$  together with a group automorphism  $\sigma : (\mathbb{Z}_2)^m \rightarrow (\mathbb{Z}_2)^m$  such that:

- (1)  $f(g \cdot x) = \sigma(g) \cdot f(x)$  for all  $g \in (\mathbb{Z}_2)^m$  and  $x \in M_1^n$ , and
- (2)  $f$  induces the identity map on the orbit space.

Under these conditions, we also say that the *free*  $(\mathbb{Z}_2)^m$ -actions on  $M_1^n$  and  $M_2^n$  are *equivalent*.

We can prove the following proposition as a by-product of our discussion.

**Proposition 1.5.** *Suppose  $Q^n$  is a small cover over a simple polytope  $P^n$  and  $M^n$  is a principal  $(\mathbb{Z}_2)^m$ -bundle over  $Q^n$ . If  $M^n$  is connected, then  $M^n$  must be equivalent to a partial quotient  $\mathcal{Z}_{P^n}/H$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .*

The paper is organized as follows. In section 2, we will briefly review the  $\mathbb{Z}_2$ -core of a manifold and the glue-back construction introduced in [16] and study some topological aspects of the glue-back construction. Then in section 3, we will give a proof of Theorem 1.4. In section 4, we will study real moment-angle manifolds from the viewpoint of glue-back construction and give a proof of Proposition 1.5.

## 2. GLUE-BACK CONSTRUCTION

Suppose  $(\mathbb{Z}_2)^m$  acts freely and smoothly on an  $n$ -dimensional closed manifold  $M^n$ . Then the orbit space  $Q^n = M^n/(\mathbb{Z}_2)^m$  is naturally a closed manifold. In the rest of this section, we always assume that  $Q^n$  is connected and  $H^1(Q^n, \mathbb{Z}_2) \neq 0$ . Indeed, if  $Q^n$  is not connected, we can just apply our discussion to each connected component of  $Q^n$ . And if  $H^1(Q^n, \mathbb{Z}_2) = 0$ ,  $M^n$  must be homeomorphic to  $Q^n \times (\mathbb{Z}_2)^m$ .

Let  $\pi : M^n \rightarrow Q^n$  be the orbit map. If we think of  $M^n$  as a principal  $(\mathbb{Z}_2)^m$ -bundle over  $Q^n$ , it is classified by an element  $\Lambda_\pi \in H^1(Q^n, (\mathbb{Z}_2)^m)$ . To recover the  $M^n$  from  $Q^n$ , we shall construct a manifold with corners from  $Q^n$  that can carry the information of  $\Lambda_\pi$ . This is done in the following way (see [16]).

By a standard argument of the intersection theory in differential topology, we can show that there exists a collection of  $(n-1)$ -dimensional compact embedded submanifolds  $\Sigma_1, \dots, \Sigma_k$  in  $Q^n$  such that their homology classes  $\{[\Sigma_1], \dots, [\Sigma_k]\}$  form a basis of  $H_{n-1}(Q^n, \mathbb{Z}_2) \cong H^1(Q^n, \mathbb{Z}_2) \neq 0$ . Moreover, we can put  $\Sigma_1, \dots, \Sigma_k$  in *general position* in  $Q^n$ , which means:

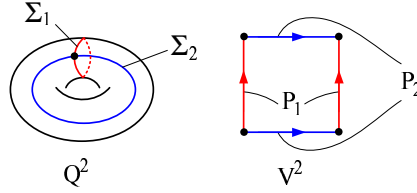
- (1)  $\Sigma_1, \dots, \Sigma_k$  intersect transversely with each other, and
- (2) if  $\Sigma_{i_1} \cap \dots \cap \Sigma_{i_s}$  is not empty, then it is an embedded submanifold of  $Q^n$  with codimension  $s$ .

Then we cut  $Q^n$  open along  $\Sigma_1, \dots, \Sigma_k$ , i.e. we choose a small tubular neighborhood  $N(\Sigma_i)$  of each  $\Sigma_i$  and remove the interior of each  $N(\Sigma_i)$  from  $Q^n$ . Then we get a nice manifold with corners  $V^n = Q^n - \bigcup_i \text{int}(N(\Sigma_i))$ , which is called a  $\mathbb{Z}_2$ -core of  $Q^n$  from cutting  $Q^n$  open along  $\Sigma_1, \dots, \Sigma_k$  (see Figure 1 for example). Recall that a manifold with corners is called *nice* if each codimension  $l$  face of the manifold is the intersection of exactly  $l$  facets (see [19] and [20]). Here, the niceness of  $V^n$  follows from the general position of  $\Sigma_1, \dots, \Sigma_k$  in  $Q^n$ . The boundary of  $N(\Sigma_i)$  is called the *cut section* of  $\Sigma_i$  in  $Q^n$ , and  $\{\Sigma_1, \dots, \Sigma_k\}$  is called a  $\mathbb{Z}_2$ -cut system of  $Q^n$ . Moreover, we can choose each  $\Sigma_i$  to be connected.

Notice that the projection  $\eta_i : \partial N(\Sigma_i) \rightarrow \Sigma_i$  is a double cover, either trivial or nontrivial. Let  $\bar{\tau}_i$  be the generator of the deck transformation of  $\eta_i$ . Then  $\bar{\tau}_i$  is a free involution on  $\partial N(\Sigma_i)$ , i.e.  $\bar{\tau}_i$  is a homeomorphism with no fixed point and  $\bar{\tau}_i^2 = \text{id}$ . By applying some local deformations to these  $\bar{\tau}_i$  if necessary (see [16]), we can construct an *involutive panel structure* on  $\partial V^n$ , which means that the boundary of  $V^n$  is the union of some compact subsets  $P_1, \dots, P_k$  (called *panels*) that satisfy the following three conditions:

- (a) each panel  $P_i$  is a disjoint union of facets of  $V^n$  and each facet is contained in exactly one panel;
- (b) there exists a free involution  $\tau_i$  on each  $P_i$  which sends a face  $f \subset P_i$  to a face  $f' \subset P_i$  (it is possible that  $f' = f$ );
- (c) for  $\forall i \neq j$ ,  $\tau_i(P_i \cap P_j) \subset P_i \cap P_j$  and  $\tau_i \circ \tau_j = \tau_j \circ \tau_i : P_i \cap P_j \rightarrow P_i \cap P_j$ .

Indeed, the  $P_i$  above consists of those facets of  $V^n$  that lie in the cut section of  $\Sigma_i$  and  $\tau_i : P_i \rightarrow P_i$  is the restriction of the modified  $\bar{\tau}_i$  to  $P_i$  (see [16] for the details of these constructions).

FIGURE 1. A  $\mathbb{Z}_2$ -core of torus

**Remark 2.1.** A more general notion of *involutive panel structure* is introduced in [16] where the involution  $\tau_i$  in (b) above is not required to be free. This general notion is used in [16] to unify the construction of all locally standard  $(\mathbb{Z}_2)^m$ -actions on closed manifolds from the orbit spaces.

Let  $\mathcal{P}(V^n) = \{P_1, \dots, P_k\}$  denote the set of all panels in  $V^n$ . Any map  $\lambda : \mathcal{P}(V^n) \rightarrow (\mathbb{Z}_2)^m$  is called a  $(\mathbb{Z}_2)^m$ -coloring on  $V^n$ , and any element in  $(\mathbb{Z}_2)^m$  is called a *color*.

Now, let us see how to recover the principal  $(\mathbb{Z}_2)^m$ -bundle  $\pi : M^n \rightarrow Q^n$  from the  $\mathbb{Z}_2$ -core  $V^n$  of  $Q^n$  and the element

$$\Lambda_\pi \in H^1(Q^n, (\mathbb{Z}_2)^m) \cong \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m). \quad (2)$$

By the Poincaré duality for closed manifolds, there is a group isomorphism

$$\kappa : H_{n-1}(Q^n, \mathbb{Z}_2) \rightarrow H_1(Q^n, \mathbb{Z}_2).$$

So we can assign an element of  $(\mathbb{Z}_2)^m$  to each panel  $P_i$  of  $V^n$  by:

$$\lambda_\pi(P_i) = \Lambda_\pi(\kappa([\Sigma_i])) \in (\mathbb{Z}_2)^m$$

We call  $\lambda_\pi$  the *associated  $(\mathbb{Z}_2)^m$ -coloring* of  $\pi : M^n \rightarrow Q^n$  on  $V^n$ .

Generally, for any  $(\mathbb{Z}_2)^m$ -coloring  $\lambda$  on  $V^n$ , we can glue  $2^m$  copy of  $V^n$  by:

$$M(V^n, \{P_i, \tau_i\}, \lambda) := V^n \times (\mathbb{Z}_2)^m / \sim \quad (3)$$

Where  $(x, g) \sim (x', g')$  whenever  $x' = \tau_i(x)$  for some  $P_i$  and  $g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m$ .

Note that if  $x$  is in the relative interior of  $P_{i_1} \cap \dots \cap P_{i_s}$ ,  $(x, g) \sim (x', g')$  if and only if  $(x', g') = (\tau_{i_s}^{\varepsilon_s} \circ \dots \circ \tau_{i_1}^{\varepsilon_1}(x), g + \varepsilon_1 \lambda(P_1) + \dots + \varepsilon_s \lambda(P_s))$  where  $\varepsilon_j = 0$  or  $1$  for any  $1 \leq j \leq s$  and  $\tau_{i_j}^0 := id$ .

$M(V^n, \{P_i, \tau_i\}, \lambda)$  is called the *glue-back construction* from  $(V^n, \lambda)$ . Also, we use  $M(V^n, \lambda)$  to denote  $M(V^n, \{P_i, \tau_i\}, \lambda)$  if there is no ambivalence about the involutive panel structure on  $V^n$  in the context.

In addition, let  $[(x, g)] \in M(V^n, \lambda)$  denote the equivalence class of  $(x, g)$  defined in (3). It is shown in [16] that  $M(V^n, \lambda)$  is a closed manifold with a smooth free

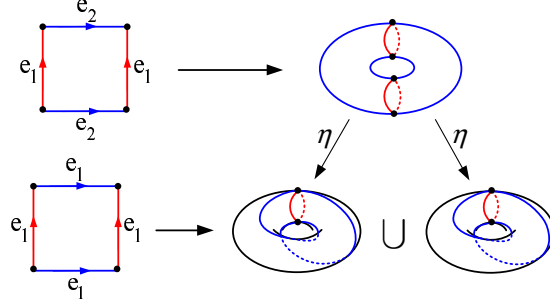


FIGURE 2.

$(\mathbb{Z}_2)^m$ -action defined by:

$$g \cdot [(x, g_0)] := [(x, g + g_0)], \quad \forall x \in V^n, \quad \forall g, g_0 \in (\mathbb{Z}_2)^m. \quad (4)$$

And the orbit space of  $M(V^n, \lambda)$  under this free  $(\mathbb{Z}_2)^m$ -action is homeomorphic to  $Q^n$ . We say (4) defines the *natural*  $(\mathbb{Z}_2)^m$ -action on  $M(V^n, \lambda)$ . In this paper, we always associate this natural free  $(\mathbb{Z}_2)^m$ -action to  $M(V^n, \lambda)$ . Moreover, for any subgroup  $N \subset (\mathbb{Z}_2)^m$ , the induced action of  $(\mathbb{Z}_2)^m/N$  on  $M(V^n, \lambda)/N$  from the natural action is also free and its orbit space is homeomorphic to  $M(V^n, \lambda)/(\mathbb{Z}_2)^m = Q^n$ . By abusing of terminology, we also call this  $(\mathbb{Z}_2)^m/N$ -action on  $M(V^n, \lambda)/N$  *natural*.

**Theorem 2.2** (Yu [16]). *For any principal  $(\mathbb{Z}_2)^m$ -bundle  $\pi : M^n \rightarrow Q^n$ , let  $\lambda_\pi$  be the associated  $(\mathbb{Z}_2)^m$ -coloring on  $V^n$ . Then  $M(V^n, \lambda_\pi)$  and  $M^n$  are equivalent as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .*

**Example 1.** Figure 2 shows two principal  $(\mathbb{Z}_2)^2$ -bundles over  $T^2$  via glue-back constructions from two different  $(\mathbb{Z}_2)^2$ -colorings on a  $\mathbb{Z}_2$ -core of  $T^2$ . The  $\{e_1, e_2\}$  in the picture is a linear basis of  $(\mathbb{Z}_2)^2$ . The first  $(\mathbb{Z}_2)^2$ -coloring gives a torus, and the second one gives a disjoint union of two tori. In addition, there is a double covering  $\eta$  (defined in (6) later) from the torus on the top to either one of the torus below it.

**Example 2.** Figure 3 shows a  $\mathbb{Z}_2$ -core of the Klein bottle with three different  $\mathbb{Z}_2$ -colorings, where  $\mathbb{Z}_2 = \langle a \rangle$ . So from the glue-back construction, we get three inequivalent double coverings of the Klein bottle. From left to right in Figure 3, the first  $\mathbb{Z}_2$ -coloring gives a torus, while the second and the third both give the Klein bottle.

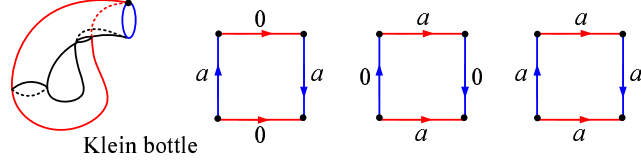


FIGURE 3.

For any integer  $m \geq 1$ , we define

$$\begin{aligned} \text{Col}_m(V^n) &:= \text{the set of all } (\mathbb{Z}_2)^m\text{-colorings on } V^n \\ &= \{\lambda \mid \lambda : \mathcal{P}(V^n) \rightarrow (\mathbb{Z}_2)^m\}, \\ L_\lambda &:= \text{the subgroup of } (\mathbb{Z}_2)^m \text{ generated by } \{\lambda(P_1), \dots, \lambda(P_k)\}, \\ \text{rank}(\lambda) &:= \dim_{\mathbb{Z}_2} L_\lambda, \quad \forall \lambda \in \text{Col}_m(V^n) \end{aligned}$$

**Theorem 2.3** (Yu [16]). *For any  $(\mathbb{Z}_2)^m$ -coloring  $\lambda$  on the panels of  $V^n$ ,  $M(V^n, \lambda)$  has  $2^{m-\text{rank}(\lambda)}$  connected components which are pairwise homeomorphic. Let  $\theta_\lambda : V^n \times (\mathbb{Z}_2)^m \rightarrow M(V^n, \lambda)$  be the quotient map. Then each connected component of  $M(V^n, \lambda)$  is homeomorphic to  $\theta_\lambda(V^n \times L_\lambda)$ , and there is a free action of  $L_\lambda \cong (\mathbb{Z}_2)^{\text{rank}(\lambda)}$  on each connected component of  $M(V^n, \lambda)$  whose orbit space is  $Q^n$ .*

In addition,  $\lambda \in \text{Col}_m(V^n)$  is called *maximally independent* if  $\text{rank}(\lambda) = k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ . Note that if  $\lambda \in \text{Col}_m(V^n)$  is maximally independent, we must have  $m \geq k$ .

**Lemma 2.4.** *For any  $m \geq \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ , if  $\lambda_1, \lambda_2 \in \text{Col}_m(V^n)$  are both maximally independent,  $M(V^n, \lambda_1)$  must be equivalent to  $M(V^n, \lambda_2)$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .*

*Proof.* Since  $\lambda_1$  and  $\lambda_2$  are both maximally independent,  $\{\lambda_1(P_1), \dots, \lambda_1(P_k)\}$  and  $\{\lambda_2(P_1), \dots, \lambda_2(P_k)\}$  are linearly independent subsets of  $(\mathbb{Z}_2)^m$ . So there exists a group automorphism  $\phi$  of  $(\mathbb{Z}_2)^m$  so that  $\phi(\lambda_1(P_i)) = \lambda_2(P_i)$  for each  $1 \leq i \leq k$ . Define a homeomorphism  $\Phi : V^n \times (\mathbb{Z}_2)^m \rightarrow V^n \times (\mathbb{Z}_2)^m$  by

$$\Phi(x, g) = (x, \phi(g)), \quad \forall x \in V^n \text{ and } g \in (\mathbb{Z}_2)^m.$$

Let  $\theta_{\lambda_i} : V^n \times (\mathbb{Z}_2)^m \rightarrow M(V^n, \lambda_i)$  ( $i = 1, 2$ ) be the quotient map defined in (3). Then obviously  $\theta_{\lambda_1}(x, g) = \theta_{\lambda_1}(x', g')$  if and only if  $\theta_{\lambda_2}(\Phi(x, g)) = \theta_{\lambda_2}(\Phi(x', g'))$ . So  $\Phi$  induces a homeomorphism  $\tilde{\Phi}$  from  $M(V^n, \lambda_1)$  to  $M(V^n, \lambda_2)$  by:

$$\tilde{\Phi}(\theta_{\lambda_1}(x, g)) = \theta_{\lambda_2}(\Phi(x, g)).$$

Moreover,  $\tilde{\Phi}$  relates the natural  $(\mathbb{Z}_2)^m$ -actions on  $M(V^n, \lambda_1)$  and  $M(V^n, \lambda_2)$  by:

$$\tilde{\Phi}(g' \cdot \theta_{\lambda_1}(x, g)) = \phi(g') \cdot \tilde{\Phi}(\theta_{\lambda_1}(x, g)), \quad \forall g' \in (\mathbb{Z}_2)^m.$$

In addition, it is easy to see that  $\tilde{\Phi}$  induces the identity map on the orbit space  $Q^n$ . So by the definition,  $M(V^n, \lambda_1)$  and  $M(V^n, \lambda_2)$  are equivalent as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .  $\square$

**Lemma 2.5.** *Suppose  $M_1$  and  $M_2$  are two principal  $(\mathbb{Z}_2)^k$ -bundles over  $Q^n$ , where  $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ . If  $M_1$  and  $M_2$  are both connected,  $M_1$  must be equivalent to  $M_2$ .*

*Proof.* By the notations in the above discussion and Theorem 2.2, we have

$$M_i \cong M(V^n, \lambda_i) \text{ for some } \lambda_i \in \text{Col}_k(V^n), \ i = 1, 2$$

In addition, since  $M_1$  and  $M_2$  are both connected, Theorem 2.3 implies that  $\text{rank}(\lambda_1) = \text{rank}(\lambda_2) = k$ , i.e.  $\lambda_1$  and  $\lambda_2$  are both maximally independent. So by Lemma 2.4,  $M(V^n, \lambda_1)$  and  $M(V^n, \lambda_2)$  are equivalent as principal  $(\mathbb{Z}_2)^k$ -bundles over  $Q^n$ .  $\square$

Next, let us study some relations between  $M(V^n, \lambda)$  for different  $\lambda \in \text{Col}_m(V^n)$ . For the sake of conciseness, for any topological space  $B$  and any field  $\mathbb{F}$ , we define

$$\text{hrk}(B, \mathbb{F}) := \sum_{i=0}^{\infty} \dim_{\mathbb{F}} H^i(B, \mathbb{F}).$$

**Lemma 2.6.** *For any double covering  $\xi : \tilde{B} \rightarrow B$  and  $\forall i \geq 0$ ,  $\dim_{\mathbb{Z}_2} H^i(\tilde{B}, \mathbb{Z}_2) \leq 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2)$ . So  $\text{hrk}(\tilde{B}, \mathbb{Z}_2) \leq 2 \cdot \text{hrk}(B, \mathbb{Z}_2)$ .*

*Proof.* The Gysin sequence of  $\xi : \tilde{B} \rightarrow B$  in  $\mathbb{Z}_2$ -coefficient reads:

$$\cdots \longrightarrow H^{i-1}(B, \mathbb{Z}_2) \xrightarrow{\phi_{i-1}} H^i(B, \mathbb{Z}_2) \xrightarrow{\xi^*} H^i(\tilde{B}, \mathbb{Z}_2) \longrightarrow H^i(B, \mathbb{Z}_2) \xrightarrow{\phi_i} \cdots$$

where  $e \in H^1(B)$  is the Euler class (or first Stiefel-Whitney class) of  $\tilde{B}$ , and  $\phi_i(\gamma) = \gamma \cup e$ ,  $\forall \gamma \in H^i(B, \mathbb{Z}_2)$ . Then by the exactness of the Gysin sequence,

$$\begin{aligned} \dim_{\mathbb{Z}_2} H^i(\tilde{B}, \mathbb{Z}_2) &= \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_{i-1}) + \dim_{\mathbb{Z}_2} \ker(\phi_i) \\ &= 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_{i-1}) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_i) \\ &\leq 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2) \end{aligned}$$

$\square$

**Remark 2.7.** In Lemma 2.6, if we replace the  $\mathbb{Z}_2$ -coefficient by  $\mathbb{Z}_p$  ( $p$  is an odd prime) or rational coefficient, the conclusion in the lemma might fail in some cases.

For any panel  $P_j \subset \mathcal{P}(V^n)$ , we define the following space which will play an important role later.

$$M_{\setminus P_j}(V^n, \lambda) := V^n \times (\mathbb{Z}_2)^m / \sim_{P_j} \quad (5)$$

where  $(x, g) \sim_{P_j} (x', g')$  whenever  $x' = \tau_i(x)$  for some  $P_i \neq P_j$  and  $g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m$ . In other words,  $M_{\setminus P_j}(V^n, \lambda)$  is the quotient space of  $V^n \times (\mathbb{Z}_2)^m$  under the rule in (3) except that we leave the interior of those facets in  $P_j \times (\mathbb{Z}_2)^m$  open. We call  $M_{\setminus P_j}(V^n, \lambda)$  a *partial glue-back* from  $(V^n, \lambda)$ . Let the corresponding quotient map be  $\theta_\lambda^{\setminus P_j} : V^n \times (\mathbb{Z}_2)^m \rightarrow M_{\setminus P_j}(V^n, \lambda)$ . Then  $\theta_\lambda^{\setminus P_j}(P_j \times (\mathbb{Z}_2)^m)$  is the boundary of  $M_{\setminus P_j}(V^n, \lambda)$ .

**Lemma 2.8.** *Suppose  $\lambda_{max} \in \text{Col}_k(V^n)$  is a maximally independent  $(\mathbb{Z}_2)^k$ -coloring on  $V^n$ , where  $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ . Then for any  $\lambda \in \text{Col}_k(V^n)$ ,*

$$\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq \text{hrk}(M(V^n, \lambda_{max}), \mathbb{Z}_2).$$

*Proof.* Without loss of generality, suppose  $\{\lambda(P_1), \dots, \lambda(P_s)\}$  is a  $\mathbb{Z}_2$ -linear basis of  $L_\lambda$ . Choose  $\omega_1, \dots, \omega_{k-s} \in (\mathbb{Z}_2)^k$  so that  $\{\lambda(P_1), \dots, \lambda(P_s), \omega_1, \dots, \omega_{k-s}\}$  forms a  $\mathbb{Z}_2$ -linear basis of  $(\mathbb{Z}_2)^k$ . Then we define a sequence of  $(\mathbb{Z}_2)^k$ -colorings  $\lambda_0, \dots, \lambda_{k-s}$  on  $V^n$  as following. For any  $0 \leq j \leq k-s$ , let

$$\lambda_j(P_i) := \begin{cases} \lambda(P_i), & 1 \leq i \leq s \text{ or } s+j < i \leq k; \\ \omega_{i-s}, & s+1 \leq i \leq s+j. \end{cases}$$

Obviously,  $\lambda_0 = \lambda$ ,  $L_\lambda = L_{\lambda_0} \subset L_{\lambda_1} \subset \dots \subset L_{\lambda_{k-s}} = (\mathbb{Z}_2)^k$  and  $\dim_{\mathbb{Z}_2}(L_{\lambda_{j+1}}) = \dim_{\mathbb{Z}_2}(L_{\lambda_j}) + 1$ . So  $\lambda_{k-s} \in \text{Col}_k(V^n)$  is maximally independent. By Lemma 2.4,  $\text{hrk}(M(V^n, \lambda_{max}), \mathbb{Z}_2) = \text{hrk}(M(V^n, \lambda_{k-s}), \mathbb{Z}_2)$ . Then it suffices to show that  $\text{hrk}(M(V^n, \lambda_{j-1}), \mathbb{Z}_2) \geq \text{hrk}(M(V^n, \lambda_j), \mathbb{Z}_2)$  for any  $1 \leq j \leq k-s$ .

Notice that the only difference between  $\lambda_j$  and  $\lambda_{j-1}$  is that:  $\lambda_j(P_{s+j}) = \omega_j$  while  $\lambda_{j-1}(P_{s+j}) = \lambda(P_{s+j})$ . So  $L_{\lambda_j} = L_{\lambda_{j-1}} \oplus \langle \omega_j \rangle \subset (\mathbb{Z}_2)^k$ . Let  $\theta_j : V^n \times (\mathbb{Z}_2)^k \rightarrow M(V^n, \lambda_j)$  be the quotient map defined by (3) for each  $j$ .

For a fixed  $j$ , let  $\tilde{K}$  and  $K$  be a connected component of  $M(V^n, \lambda_j)$  and  $M(V^n, \lambda_{j-1})$  respectively. By Theorem 2.3, we can assume that:

$$\tilde{K} = \theta_j(V^n \times L_{\lambda_j}), \quad K = \theta_{j-1}(V^n \times L_{\lambda_{j-1}}).$$

Next, we define a free involution  $\eta$  on  $\tilde{K}$  by: for any  $[(x, g)] \in \tilde{K}$ ,

$$\eta([(x, g)]) = (\lambda(P_{s+j}) + \omega_j) \cdot [(x, g)] \stackrel{(4)}{=} [(x, g + \lambda(P_{s+j}) + \omega_j)]. \quad (6)$$

**Claim:** the orbit space of  $\tilde{K}$  under the free involution  $\eta$  is homeomorphic to  $K$ . So  $\tilde{K}$  is a double covering of  $K$  (see Example 1).

To prove the claim, first let  $\theta_j^{\setminus P_{s+j}} : V^n \times (\mathbb{Z}_2)^m \rightarrow M_{\setminus P_{s+j}}(V^n, \lambda_j)$  be the quotient map of the partial glue-back defined by (5). For any  $(x, g) \in V^n \times (\mathbb{Z}_2)^m$ , denote  $\overline{(x, g)} := \theta_j^{\setminus P_{s+j}}(x, g)$ . And we define

$$W^n := \theta_j^{\setminus P_{s+j}}(V^n \times L_{\lambda_j}).$$

Geometrically,  $W^n$  is the quotient space of  $V^n \times L_{\lambda_j}$  under  $\theta_j$  except that we do not glue those facets in  $P_{s+j} \times L_{\lambda_j}$ . By the definition,  $W^n = W_0^n \cup W_1^n$  where

$$W_0^n = \theta_j^{\setminus P_{s+j}}(V^n \times L_{\lambda_{j-1}}), \quad W_1^n = \theta_j^{\setminus P_{s+j}}(V^n \times (L_{\lambda_{j-1}} + \omega_j)).$$

Let  $A_0 = \theta_j^{\setminus P_{s+j}}(P_{s+j} \times L_{\lambda_{j-1}}) \subset \partial W_0^n$ ,  $A_1 = \theta_j^{\setminus P_{s+j}}(P_{s+j} \times (L_{\lambda_{j-1}} + \omega_j)) \subset \partial W_1^n$ .

Here, the fact that  $\omega_j$  is linearly independent from  $L_{\lambda_{j-1}}$  is essential for these constructions. Otherwise,  $W_0^n$  and  $W_1^n$  would be the same space.

It is easy to see that  $\tilde{K}$  is the gluing of  $W_0^n$  and  $W_1^n$  by a homeomorphism  $\varphi : A_0 \rightarrow A_1$  defined by: for  $\forall x_0 \in P_{s+j}$  and  $\forall g_0 \in L_{\lambda_{j-1}}$ ,

$$\overline{(x_0, g_0)} \in A_0 \xrightarrow{\varphi} \overline{(\tau_{s+j}(x_0), g_0 + \omega_j)} \in A_1.$$

Let  $p : W^n = W_0^n \cup W_1^n \rightarrow W_0^n \cup_{\varphi} W_1^n = \tilde{K}$  denote this quotient map. So by our notations,  $p(\overline{(x, g)}) = [(x, g)]$  for any  $\overline{(x, g)} \in W^n$ .

Obviously, we have  $\tilde{K} = p(W_0^n) \cup p(W_1^n)$  and  $p(W_0^n) \cap p(W_1^n) = p(A_0) = p(A_1)$ . The key observation here is that the involution  $\eta$  maps  $p(W_0^n)$  homeomorphically to  $p(W_1^n)$ , and the action of  $\eta$  on  $p(A_0) = p(A_1)$  is: for any  $\overline{(x_0, g_0)} \in A_0$ ,

$$\begin{aligned} \eta(p(\overline{(x_0, g_0)})) &= \eta([(x_0, g_0)]) = \eta([\tau_{s+j}(x_0), g_0 + \omega_j]) \\ &\stackrel{(6)}{=} [(\tau_{s+j}(x_0), g_0 + \lambda(P_{s+j}))] = p(\overline{(\tau_{s+j}(x_0), g_0 + \lambda(P_{s+j}))}) \end{aligned}$$

So the orbit space of  $\tilde{K}$  under the action of  $\eta$  is homeomorphic to the quotient space of  $W_0^n$  by identifying its boundary point  $\overline{(x_0, g_0)} \in A_0$  with another point  $\overline{(\tau_{s+j}(x_0), g_0 + \lambda(P_{s+j}))} \in A_0$ , which is exactly the same as  $\theta_{j-1}(V^n \times L_{\lambda_{j-1}}) = K$  (see Example 1). So our claim is proved.

Then by Lemma 2.6,  $\text{hrk}(\tilde{K}, \mathbb{Z}_2) \leq 2 \cdot \text{hrk}(K, \mathbb{Z}_2)$ . Moreover, by Theorem 2.3, the connected components in each  $M(V^n, \lambda_j)$  are pairwise homeomorphic and the number of connected components of  $M(V^n, \lambda_{j-1})$  is twice that of  $M(V^n, \lambda_j)$ , so we have  $\text{hrk}(M(V^n, \lambda_{j-1}), \mathbb{Z}_2) \geq \text{hrk}(M(V^n, \lambda_j), \mathbb{Z}_2)$ . The lemma is proved.  $\square$

### 3. PROOF OF THEOREM 1.4

First, we quote a lemma shown in [13]. But we will slightly rephrase the original statement of this lemma to adapt to our proof of Theorem 1.4.

**Lemma 3.1** (Ustinovsky [13]). *Let  $(X, A)$  be a pair of CW-complexes such that  $A$  has a collar neighborhood  $U(A)$  in  $X$ , that is,  $(U(A), A) \cong (A \times [0, 1], A \times 0)$ . Suppose we have a homeomorphism  $\varphi : A \rightarrow A$  which can be extended to a homeomorphism  $\tilde{\varphi} : X \rightarrow X$ . Let  $Y = X_1 \cup_{\varphi} X_2$  be the space obtained by gluing two copies of  $X$  along  $A$  via the map  $\varphi$ . Then for any field  $\mathbb{F}$ , we have:  $\text{hrk}(Y, \mathbb{F}) \geq \text{hrk}(A, \mathbb{F})$ .*

*Proof.* The argument here is almost the same as in [13]. Let  $U_1(A)$  and  $U_2(A)$  be the collar neighborhoods of  $A$  in  $X_1$  and  $X_2$  respectively. Consider an open cover  $Y = W_1 \cup W_2$  where  $W_1 = X_1 \cup U_2(A)$  and  $W_2 = X_2 \cup U_1(A)$ . Then the Mayer-Vietoris sequence of cohomology groups for this open cover reads (we omit all the coefficient  $\mathbb{F}$  below):

$$\cdots \rightarrow H^{j-1}(W_1 \cap W_2) \xrightarrow{\delta_{(j)}^*} H^j(Y) \xrightarrow{g_{(j)}^*} H^j(W_1) \oplus H^j(W_2) \xrightarrow{p_{(j)}^*} H^j(W_1 \cap W_2) \rightarrow \cdots$$

Here the map  $p_{(j)}^* = i_1^* \oplus -i_2^*$ , where  $i_1$  and  $i_2$  are inclusions of  $W_1 \cap W_2$  into  $W_1$  and  $W_2$  respectively. Since  $W_1$  and  $W_2$  are both homotopy equivalent to  $X$  and  $W_1 \cap W_2 = U_1(A) \cup U_2(A) \cong A \times (-1, 1)$ , we get another long exact sequence which is equivalent to the above one:

$$\cdots \longrightarrow H^{j-1}(A) \xrightarrow{\hat{\delta}_{(j)}^*} H^j(Y) \xrightarrow{\hat{g}_{(j)}^*} H^j(X_1) \oplus H^j(X_2) \xrightarrow{\hat{p}_{(j)}^*} H^j(A) \longrightarrow \cdots$$

Notice that the  $\hat{p}_{(j)}^* = \iota_1^* \oplus -(\iota_2 \circ \varphi)^*$  where  $\iota_1$  and  $\iota_2$  are inclusions of  $A$  into  $X_1$  and  $X_2$  respectively. For any  $\gamma \in H^j(X_1)$ , it is easy to see that  $(\gamma, (\tilde{\varphi}^{-1})^* \gamma)$  is in  $\ker(\hat{p}_{(j)}^*)$ . This implies that  $\dim \ker(\hat{p}_{(j)}^*) \geq \dim H^j(X)$  and so  $\dim \text{Im}(\hat{p}_{(j)}^*) \leq \dim H^{j-1}(A)$ . Then we have:

$$\begin{aligned} \dim H^j(Y) &= \dim \ker(\hat{g}_{(j)}^*) + \dim \text{Im}(\hat{g}_{(j)}^*) = \dim \text{Im}(\hat{\delta}_{(j)}^*) + \dim \ker(\hat{p}_{(j)}^*) \\ &\geq \dim H^{j-1}(A) - \dim \text{Im}(\hat{p}_{(j-1)}^*) + \dim H^j(X) \\ &\geq \dim H^{j-1}(A) - \dim H^{j-1}(X) + \dim H^j(X). \end{aligned}$$

By summing up these inequalities over all indices  $j$ , we get:

$$\begin{aligned} \text{hrk}(Y, \mathbb{F}) &= \sum_j \dim H^j(Y) \geq \sum_j \dim H^{j-1}(A) - \dim H^{j-1}(X) + \dim H^j(X) \\ &= \sum_j \dim H^{j-1}(A) = \text{hrk}(A, \mathbb{F}). \end{aligned}$$

□

**Remark 3.2.** In the above lemma, the assumption that  $\varphi : A \rightarrow A$  can be extended to a homeomorphism  $\tilde{\varphi} : X \rightarrow X$  is essential, otherwise the claim may not be true.

**Proof of Theorem 1.4:** We shall organize the proof by an induction on the dimension of  $M^n$ . When  $n = 1$ , since a principal  $(\mathbb{Z}_2)^m$ -bundle over a circle must be a disjoint union of  $2^m$  or  $2^{m-1}$  circles, so the theorem holds. Then we assume the theorem holds for manifolds with dimension less than  $n$ .

Suppose  $P^n$  is an  $n$ -dimensional simple convex polytope with  $k + n$  facets  $F_1, \dots, F_{k+n}$  ( $k \geq 1$ ) and  $\pi_\mu : Q^n \rightarrow P^n$  is a small cover over  $P^n$  with the characteristic function  $\mu$ . For any face  $\mathbf{f} = F_{i_1} \cap \dots \cap F_{i_l}$  of  $P^n$ , let  $G_{\mathbf{f}}^\mu$  be the rank- $l$  subgroup of  $(\mathbb{Z}_2)^n$  generated by  $\mu(F_1), \dots, \mu(F_l)$ . Then by the definition,

$$Q^n = P^n \times (\mathbb{Z}_2)^n / \sim, \quad (p, w) \sim (p', w') \iff p = p', w - w' \in G_{\mathbf{f}(p)}^\mu, \quad (7)$$

where  $\mathbf{f}(p)$  is the unique face of  $P^n$  that contains  $p$  in its relative interior. It was shown in [17] that the  $\mathbb{Z}_2$ -Betti numbers of  $Q^n$  can be computed from the  $h$ -vector of  $P^n$ . In particular,  $H_{n-1}(Q^n, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k$ .

Next, we choose an arbitrary vertex  $v_0$  of  $P^n$ . By re-indexing the facets of  $P^n$ , we can assume  $F_1, \dots, F_k$  are those facets of  $P^n$  that are not incident to  $v_0$ . Then according to [17], the homology classes of the facial submanifolds  $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$  form a  $\mathbb{Z}_2$ -linear basis of  $H_{n-1}(Q^n, \mathbb{Z}_2)$ . Cutting  $Q^n$  open along  $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$  will give us a  $\mathbb{Z}_2$ -core of  $Q^n$ , denoted by  $V^n$ . We can think of  $V^n$  as a partial gluing of the  $2^n$  copies of  $P^n$  according to the rule in (7) except that we leave the facets  $F_1, \dots, F_k$  in each copy of  $P^n$  open (see Figure 4 for example). Let  $\zeta : P^n \times (\mathbb{Z}_2)^n \rightarrow V^n$  denote the quotient map and let  $P_1, \dots, P_k$  be the panels of  $V^n$  corresponding to  $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$ . Then each  $P_i$  consists of  $2^n$  copies of  $F_i$  and the involutive panel structure  $\{\tau_i : P_i \rightarrow P_i\}_{1 \leq i \leq k}$  on  $V^n$  can be written as:

$$\tau_i(\zeta(p, w)) = \zeta(p, w + \mu(F_i)), \quad \forall p \in F_i, \forall w \in (\mathbb{Z}_2)^n \quad (8)$$

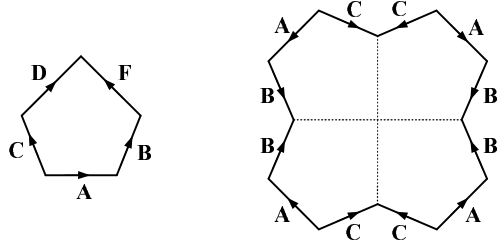
Obviously, each  $\tau_i$  extends to an automorphism  $\tilde{\tau}_i$  of  $V^n$  given by the same form:

$$\tilde{\tau}_i(\zeta(p, w)) = \zeta(p, w + \mu(F_i)), \quad \forall p \in P^n, \forall w \in (\mathbb{Z}_2)^n \quad (9)$$

And these  $\tilde{\tau}_i$  commute with each other, i.e.  $\tilde{\tau}_i \circ \tilde{\tau}_j = \tilde{\tau}_j \circ \tilde{\tau}_i$ ,  $1 \leq i, j \leq k$ . So each  $\tilde{\tau}_i$  will preserve any panel  $P_j$  of  $V^n$ .

To prove Theorem 1.4, it suffices to show that  $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$  for any  $\lambda \in \text{Col}_m(V^n)$  (because of Theorem 2.2).

First, we assume  $m = k$ . Let  $\lambda_0$  be a maximally independent  $(\mathbb{Z}_2)^k$ -coloring of  $V^n$ , i.e.  $\text{rank}(\lambda_0) = k$ . By Lemma 2.8, for  $\forall \lambda \in \text{Col}_k(V^n)$ ,  $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq$

FIGURE 4. A  $\mathbb{Z}_2$ -core of a small cover in dimension 2

$\text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2)$ . So it suffices to prove that

$$\text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k. \quad (10)$$

Indeed, the (10) follows from Theorem 1.1 and Lemma 2.5 (see the Remark 3.3 below). But here we will give another proof of (10) which only uses the Lemma 3.1 taken from [13]. Our proof will take advantage of the special symmetries of small covers (see (8) and (9)), and it is more natural from the viewpoint of the glue-back construction.

Since  $\lambda_0$  is maximally independent, by Lemma 2.4, we can assume  $\lambda_0(P_i) = e_i$ ,  $1 \leq i \leq k$ , where  $\{e_1, \dots, e_k\}$  is a linear basis of  $(\mathbb{Z}_2)^k$ . Let  $\theta_{\lambda_0} : V^n \times (\mathbb{Z}_2)^k \rightarrow M(V^n, \lambda_0)$  be the quotient map defined by (3).

Now take an arbitrary panel of  $V^n$ , say  $P_1$  and let  $M_{\setminus P_1}(V^n, \lambda_0)$  be a partial glue-back from  $(V^n, \lambda_0)$  defined by (5). Let  $\theta_{\lambda_0}^{\setminus P_1} : V^n \times (\mathbb{Z}_2)^k \rightarrow M_{\setminus P_1}(V^n, \lambda_0)$  be the corresponding quotient map. Suppose  $H$  is the subgroup of  $(\mathbb{Z}_2)^k$  generated by  $\{e_2, \dots, e_k\}$ . Then we define:

$$Y_1 = \theta_{\lambda_0}^{\setminus P_1}(V^n \times H), \quad Y_2 = \theta_{\lambda_0}^{\setminus P_1}(V^n \times (e_1 + H)); \quad (11)$$

$$A_1 = \theta_{\lambda_0}^{\setminus P_1}(P_1 \times H), \quad A_2 = \theta_{\lambda_0}^{\setminus P_1}(P_1 \times (e_1 + H)). \quad (12)$$

Obviously,  $A_1 = \partial Y_1$ ,  $A_2 = \partial Y_2$  and there is homeomorphism  $\Pi : Y_1 \rightarrow Y_2$  with  $\Pi(A_1) = A_2$ . Indeed,  $\Pi$  is given by:

$$\Pi(\theta_{\lambda_0}^{\setminus P_1}(x, h)) = \theta_{\lambda_0}^{\setminus P_1}(x, h + e_1), \quad \forall x \in V^n, \forall h \in H.$$

It is easy to see that  $M(V^n, \lambda_0)$  is the gluing of  $Y_1$  and  $Y_2$  along their boundary by a homeomorphism  $\varphi : A_1 \rightarrow A_2$  defined by:

$$\varphi(\theta_{\lambda_0}^{\setminus P_1}(x_1, h)) = \theta_{\lambda_0}^{\setminus P_1}(\tau_1(x_1), h + e_1), \quad \forall x_1 \in P_1, \forall h \in H.$$

Moreover, since  $\tau_1 : P_1 \rightarrow P_1$  extends to a homeomorphism  $\tilde{\tau}_1 : V^n \rightarrow V^n$  (see (8) and (9)), we can extend  $\varphi$  to a homeomorphism  $\tilde{\varphi} : Y_1 \rightarrow Y_2$  by:

$$\tilde{\varphi}(\theta_{\lambda_0}^{\setminus P_1}(x, h)) = \theta_{\lambda_0}^{\setminus P_1}(\tilde{\tau}_1(x), h + e_1), \quad \forall x \in V^n, \forall h \in H.$$

The  $\tilde{\varphi}$  is well-defined since  $\tilde{\tau}_1$  commutes with each  $\tau_i$  on  $P_i$  (see (3) and (9)).

Now, if we identify  $(Y_1, A_1)$  with  $(Y_2, A_2)$  via  $\Pi$ , we get a decomposition of  $M(V^n, \lambda_0)$  that satisfies all the conditions in Lemma 3.1. So Lemma 3.1 implies:

$$\mathrm{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq \mathrm{hrk}(A_1, \mathbb{Z}_2). \quad (13)$$

In addition, let  $q : Y_1 \cup Y_2 \rightarrow M(V^n, \lambda_0)$  be the quotient map. It is easy to see that:  $A_1 \cong q(A_1) = \theta_{\lambda_0}^{-1}(\Sigma_1)$ . Since  $\theta_{\lambda_0}^{-1}(\Sigma_1)$  is a principal  $(\mathbb{Z}_2)^k$ -bundle over  $\Sigma_1$  and  $\Sigma_1$  is a small cover over  $F_1$  with dimension  $n - 1$ , so by the induction hypothesis, we have  $\mathrm{hrk}(\theta_{\lambda_0}^{-1}(\Sigma_1), \mathbb{Z}_2) \geq 2^k$ . So  $\mathrm{hrk}(A_1, \mathbb{Z}_2) = \mathrm{hrk}(\theta_{\lambda_0}^{-1}(\Sigma_1), \mathbb{Z}_2) \geq 2^k$ . Then by (13), we get  $\mathrm{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k$ . So this case is confirmed.

Next, we assume  $m < k$ . Let  $\iota : (\mathbb{Z}_2)^m \hookrightarrow (\mathbb{Z}_2)^k$  be the standard inclusion and define  $\hat{\lambda} := \iota \circ \lambda$ . We consider  $\hat{\lambda}$  as a  $(\mathbb{Z}_2)^k$ -coloring on  $V^n$ . So by the above argument,  $\mathrm{hrk}(M(V^n, \hat{\lambda}), \mathbb{Z}_2) \geq 2^k$ . Since by Theorem 2.3,  $M(V^n, \hat{\lambda})$  consists of  $2^{k-m}$  copies of  $M(V^n, \lambda)$ , so  $\mathrm{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$ .

Finally, we assume  $m > k$ . Since  $\mathrm{rank}(\lambda) \leq k$ , with a proper change of basis, we can assume  $L_\lambda \subseteq (\mathbb{Z}_2)^k \subset (\mathbb{Z}_2)^m$ . Let  $\varrho : (\mathbb{Z}_2)^m \rightarrow (\mathbb{Z}_2)^k$  be the standard projection. Define  $\bar{\lambda} := \varrho \circ \lambda$ . Similarly, we consider  $\bar{\lambda}$  as a  $(\mathbb{Z}_2)^k$ -coloring on  $V^n$  and so we have  $\mathrm{hrk}(M(V^n, \bar{\lambda}), \mathbb{Z}_2) \geq 2^k$ . Since by Theorem 2.3,  $M(V^n, \bar{\lambda})$  consists of  $2^{m-k}$  copies of  $M(V^n, \lambda)$ , so  $\mathrm{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$ .

So for  $\forall m \geq 1$  and  $\forall \lambda \in \mathrm{Col}_m(V^n)$ , we always have  $\mathrm{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$ . The induction is completed.  $\square$

**Remark 3.3.** Notice that  $M(V^n, \lambda_0)$  is a connected principal  $(\mathbb{Z}_2)^k$ -bundle over  $Q^n$ , so is the real moment-angle manifold  $\mathcal{Z}_{P^n}$ . Then by Lemma 2.5,  $M(V^n, \lambda_0)$  is homeomorphic to  $\mathcal{Z}_{P^n}$ . Then the result of the Theorem 1.1 also tells us that  $\mathrm{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k$ .

A crucial observation in the above proof is that: when  $\lambda_0 \in \mathrm{Col}_k(V^n)$  is maximally independent, we can always get the type of decomposition of  $M(V^n, \lambda_0)$  as in Lemma 3.1, which allows us to use the induction hypothesis. However, for an arbitrary  $\lambda \in \mathrm{Col}_k(V^n)$ , this type of decomposition for  $M(V^n, \lambda)$  may not exist (at least not very obvious).

For example, in the lower picture in Figure 2, we have a principal  $(\mathbb{Z}_2)^2$ -bundle  $\pi : M^2 \rightarrow T^2$  where  $M^2$  is a disjoint union of two tori. The union of the two meridians in  $M^2$  is the inverse image of a meridian in  $T^2$  under  $\pi$ . If we cut  $M^2$  open along these two meridians, we will get two circular cylinders. But  $M^2$  here is not got by gluing these two cylinders together. This is because that the colors of the  $(\mathbb{Z}_2)^2$ -coloring on the two panels are not linearly independent. So

the construction in (11) for this case fails to give us the type of decomposition of  $M^2$  as in Lemma 3.1.

So when  $\lambda \in \text{Col}_k(V^n)$  is not maximally independent, we may not be able to directly apply the induction hypothesis to  $M(V^n, \lambda)$  as we do to  $M(V^n, \lambda_0)$  above. But these cases are settled by Lemma 2.8.

#### 4. REAL MOMENT-ANGLE MANIFOLDS FROM THE VIEWPOINT OF GLUE-BACK CONSTRUCTION

Suppose  $P^n$  is an  $n$ -dimensional simple polytope with  $k+n$  facets  $F_1, \dots, F_{k+n}$  and  $\pi_\mu : Q^n \rightarrow P^n$  is a small cover with a characteristic function  $\mu$  on  $P^n$ . We know that  $\mu : \{F_1, \dots, F_{k+n}\} \rightarrow (\mathbb{Z}_2)^n$  satisfies: whenever  $F_{i_1} \cap \dots \cap F_{i_s} \neq \emptyset$ ,  $\lambda(F_{i_1}), \dots, \lambda(F_{i_s})$  are linearly independent vectors in  $(\mathbb{Z}_2)^n$ . For the convenience of our following discussion, let a linear basis of  $(\mathbb{Z}_2)^n$  be  $\{e_{k+1}, \dots, e_{k+n}\}$ .

In this section, we will use the  $\mathbb{Z}_2$ -core  $V^n$  of  $Q^n$  as described at the beginning of the proof of Theorem 1.4, whose involutive panel structure  $\{\tau_i : P_i \rightarrow P_i\}_{1 \leq i \leq k}$  is defined by (8).

It is well known that the real moment-angle manifold  $\mathcal{Z}_{P^n}$  is a principal  $(\mathbb{Z}_2)^k$ -bundle over  $Q^n$ . Since  $\mathcal{Z}_{P^n}$  is connected, by Lemma 2.5,  $\mathcal{Z}_{P^n}$  is homeomorphic to  $M(V^n, \lambda_0)$  for any maximally independent  $\lambda_0 \in \text{Col}_k(V^n)$ . Now, let us compare the definitions of  $\mathcal{Z}_{P^n}$  and  $M(V^n, \lambda_0)$  and then construct a homeomorphism from  $M(V^n, \lambda_0)$  to  $\mathcal{Z}_{P^n}$  explicitly.

Suppose  $\{e_1, \dots, e_k\}$  is a linear basis of  $(\mathbb{Z}_2)^k$ . We choose  $\lambda_0(P_i) = e_i$  for  $1 \leq i \leq k$ . Let  $\theta_{\lambda_0} : V^n \times (\mathbb{Z}_2)^k \rightarrow M(V^n, \lambda_0)$  be the quotient map defined by (3). By the definition, the panel  $P_i$  consists of  $2^n$  copies of  $F_i$ , and any  $(x, g) \in P_i \times (\mathbb{Z}_2)^k$  is identified with  $(\tau_i(x), g + \lambda_0(P_i)) = (\tau_i(x), g + e_i)$  under  $\theta_{\lambda_0}$ .

Suppose  $(\mathbb{Z}_2)^{k+n} = (\mathbb{Z}_2)^k \oplus (\mathbb{Z}_2)^n$  with a linear basis  $\{e_1, \dots, e_k, e_{k+1}, \dots, e_{k+n}\}$ , and we identify  $(\mathbb{Z}_2)^k$  and  $(\mathbb{Z}_2)^n$  as a subgroup of  $(\mathbb{Z}_2)^{k+n}$  in the obvious way. The real moment-angle manifold  $\mathcal{Z}_{P^n}$  corresponds to a  $(\mathbb{Z}_2)^{k+n}$ -coloring  $\mu_0$  on  $P^n$  which is  $\mu_0(F_i) = e_i$  for any  $1 \leq i \leq k+n$ . By the definition,  $\mathcal{Z}_{P^n}$  is obtained by gluing  $2^{k+n}$  copies of  $P^n$  together by identifying any  $(p, \tilde{g}) \in F_i \times (\mathbb{Z}_2)^{k+n}$  with  $(p, \tilde{g} + \mu_0(F_i))$  for all facet  $F_i$ . Let  $\Theta : P^n \times (\mathbb{Z}_2)^{k+n} \rightarrow \mathcal{Z}_{P^n}$  be the corresponding quotient map.

To see the relationship between  $\mathcal{Z}_{P^n}$  and  $M(V^n, \lambda)$ , let us decompose the above gluing process defined by  $\Theta$  into two steps. In the first step, we glue the  $2^{k+n}$  copies of  $P^n$  only along the facets  $F_{k+1}, \dots, F_{k+n}$  on their boundaries. Then we will get  $2^k$  copies of  $V^n$ , each of which is the gluing of  $2^n$  copies of  $P^n$ . We readily index these  $V^n$ 's by the elements of  $(\mathbb{Z}_2)^k = \langle e_1, \dots, e_k \rangle \subset (\mathbb{Z}_2)^{k+n}$ . Let  $\tilde{\zeta} : P^n \times (\mathbb{Z}_2)^{k+n} \rightarrow V^n \times (\mathbb{Z}_2)^k$  denote this partial gluing map, and let

$J = (\mathbb{Z}_2)^n = \langle e_{k+1}, \dots, e_{k+n} \rangle \subset (\mathbb{Z}_2)^{k+n}$ . Then we have:

$$V^n \times g = \tilde{\zeta}(P^n \times (g + J)), \quad \forall g \in (\mathbb{Z}_2)^k. \quad (14)$$

The  $(\mathbb{Z}_2)^{k+n}$ -coloring  $\mu_0$  on  $P^n$  induces a coloring  $\hat{\lambda}_0$  on  $V^n$  valued in  $(\mathbb{Z}_2)^k \subset (\mathbb{Z}_2)^{k+n}$  by:  $\hat{\lambda}_0(P_i) = \mu_0(F_i) = e_i$ ,  $1 \leq i \leq k$ . Note that the two  $(\mathbb{Z}_2)^k$ -coloring  $\hat{\lambda}_0$  and  $\lambda_0$  on  $V^n$  actually coincide, but they are used for different purposes.

In the second step, the  $\mathcal{Z}_{P^n}$  is obtained from gluing the  $2^k$  copies of  $V^n$  by identifying any  $(x, g) \in P_i \times (\mathbb{Z}_2)^k$  with  $(x, g + \hat{\lambda}_0(P_i)) = (x, g + e_i)$  for all  $P_i$ ,  $1 \leq i \leq k$ . Let  $\vartheta_{\hat{\lambda}_0} : V^n \times (\mathbb{Z}_2)^k \rightarrow \mathcal{Z}_{P^n}$  denote this quotient map. Obviously,

$$\Theta = \vartheta_{\hat{\lambda}_0} \circ \tilde{\zeta} : P^n \times (\mathbb{Z}_2)^{k+n} \rightarrow \mathcal{Z}_{P^n}.$$

Notice that the domains of  $\theta_{\lambda_0}$  and  $\vartheta_{\hat{\lambda}_0}$  are both  $V^n \times (\mathbb{Z}_2)^k$ . By comparing their definitions, we see that the difference between  $\theta_{\lambda_0}$  and  $\vartheta_{\hat{\lambda}_0}$  is just the involution  $\tau_i$  on each panel  $P_i$ . Since each  $\tau_i$  extends to an involution  $\tilde{\tau}_i$  on  $V^n$  (see (9)), for any  $g = \sum_{i=1}^k \varepsilon_i e_i \in (\mathbb{Z}_2)^k$  where  $\varepsilon_i \in \{0, 1\}$ , we get an involution  $\psi_g : V^n \rightarrow V^n$  by:

$$\psi_g(x) := \tilde{\tau}_k^{\varepsilon_k} \circ \dots \circ \tilde{\tau}_1^{\varepsilon_1}(x), \quad \forall x \in V^n. \quad (15)$$

The  $\psi_g$  is independent of the ordering of  $\tilde{\tau}_1, \dots, \tilde{\tau}_k$  since they commute with each other. Using these involutions  $\{\psi_g\}_{g \in (\mathbb{Z}_2)^k}$ , we can define a homeomorphism  $\Psi : V^n \times (\mathbb{Z}_2)^k \rightarrow V^n \times (\mathbb{Z}_2)^k$  by:

$$\Psi(x, g) := (\psi_g(x), g), \quad \forall x \in V^n, \quad \forall g \in (\mathbb{Z}_2)^k.$$

Obviously,  $\Psi \circ \Psi = id$ . Moreover, we can show the following lemma.

**Lemma 4.1.**  $\theta_{\lambda_0}(x, g) = \theta_{\lambda_0}(x', g')$  if and only if  $\vartheta_{\hat{\lambda}_0}(\Psi(x, g)) = \vartheta_{\hat{\lambda}_0}(\Psi(x', g'))$ .

*Proof.* If  $\theta_{\lambda_0}(x, g) = \theta_{\lambda_0}(x', g')$ , without loss of generality, we can assume that

$$x \in P_i \text{ and } (x', g') = (\tau_i(x), g + \lambda_0(P_i)) = (\tau_i(x), g + e_i) \text{ for some } i.$$

Then  $\Psi(x', g') = (\psi_{g+e_i}(\tau_i(x)), g + e_i) = (\psi_g \circ \tilde{\tau}_i(\tau_i(x)), g + e_i) = (\psi_g(x), g + e_i)$ . So  $\vartheta_{\hat{\lambda}_0}(\Psi(x', g')) = \vartheta_{\hat{\lambda}_0}(\psi_g(x), g + e_i) = \vartheta_{\hat{\lambda}_0}(\psi_g(x), g) = \vartheta_{\hat{\lambda}_0}(\Psi(x, g))$ .

Conversely, if  $\vartheta_{\hat{\lambda}_0}(\Psi(x, g)) = \vartheta_{\hat{\lambda}_0}(\Psi(x', g'))$ , without loss of generality, we can assume that  $x \in P_i$  and  $\Psi(x', g') = (\psi_{g'}(x'), g') = (\psi_g(x), g + e_i)$  for some  $i$ . Then we have  $\psi_{g'}(x') = \psi_{g+e_i}(x') = \psi_g(\tilde{\tau}_i(x')) = \psi_g(x)$ . So  $\tilde{\tau}_i(x') = x$  and so  $x' = \tilde{\tau}_i(x) = \tau_i(x)$ . Therefore,  $\theta_{\lambda_0}(x', g') = \theta_{\lambda_0}(\tau_i(x), g + e_i) = \theta_{\lambda_0}(x, g)$ .  $\square$

By the above lemma,  $\Psi$  induces a homeomorphism  $\tilde{\Psi} : M(V^n, \lambda_0) \rightarrow \mathcal{Z}_{P^n}$  by:

$$\tilde{\Psi}(\theta_{\lambda_0}(x, g)) := \vartheta_{\hat{\lambda}_0}(\Psi(x, g)), \quad \forall x \in V^n, \quad \forall g \in (\mathbb{Z}_2)^k. \quad (16)$$

It is easy to see that  $\tilde{\Psi}^{-1}(\vartheta_{\hat{\lambda}_0}(x, g)) = \theta_{\lambda_0}(\Psi(x, g)) = \theta_{\lambda_0}(\psi_g(x), g)$ .

Next, let us see how  $\tilde{\Psi}$  relates the natural  $(\mathbb{Z}_2)^k$ -action on  $M(V^n, \lambda_0)$  and the canonical  $(\mathbb{Z}_2)^{k+n}$ -action on  $\mathcal{Z}_{P^n}$ . The natural action of  $(\mathbb{Z}_2)^k$  on  $M(V^n, \lambda_0)$  defined by (4) is

$$g' \cdot \theta_{\lambda_0}(x, g) = \theta_{\lambda_0}(x, g' + g), \quad \forall x \in V^n, \quad \forall g', g \in (\mathbb{Z}_2)^k. \quad (17)$$

This induces a free action of  $(\mathbb{Z}_2)^k$  on  $\mathcal{Z}_{P^n}$  through the homeomorphism  $\tilde{\Psi}$  by:

$$g' \star \vartheta_{\hat{\lambda}_0}(x, g) = \vartheta_{\hat{\lambda}_0}(\psi_{g'}(x), g' + g), \quad \forall x \in V^n, \quad \forall g', g \in (\mathbb{Z}_2)^k, \quad (18)$$

so that  $\tilde{\Psi}$  is equivariant with respect to the  $(\mathbb{Z}_2)^k$ -actions defined by (17) and (18):

$$\tilde{\Psi}(g' \cdot \theta_{\lambda_0}(x, g)) = g' \star \tilde{\Psi}(\theta_{\lambda_0}(x, g)), \quad \forall g' \in (\mathbb{Z}_2)^k. \quad (19)$$

On the other hand, the canonical  $(\mathbb{Z}_2)^{k+n}$ -action on  $\mathcal{Z}_{P^n}$  is defined by:

$$\tilde{g}_0 \circledast \Theta(p, \tilde{g}) = \Theta(p, \tilde{g}_0 + \tilde{g}), \quad \forall p \in P^n, \quad \forall \tilde{g}_0, \tilde{g} \in (\mathbb{Z}_2)^{k+n}. \quad (20)$$

We can first interpret the free  $(\mathbb{Z}_2)^k$ -action  $\star$  on  $\mathcal{Z}_{P^n}$  defined by (18) as the restriction of the canonical  $(\mathbb{Z}_2)^{k+n}$ -action  $\circledast$  on  $\mathcal{Z}_{P^n}$  to a subtorus of  $(\mathbb{Z}_2)^{k+n}$ .

In fact, by the definition of  $\tilde{\tau}_i$  in (9) and (14), if  $g' = \sum_{i=1}^k \varepsilon'_i e_i \in (\mathbb{Z}_2)^k$  where  $\varepsilon'_i \in \{0, 1\}$  for  $1 \leq i \leq k$ , the action of  $g'$  on  $\mathcal{Z}_{P^n}$  defined by (18) is:

$$g' \star \vartheta_{\hat{\lambda}_0}(\tilde{\zeta}(p, \tilde{g})) = \vartheta_{\hat{\lambda}_0}\left(\tilde{\zeta}\left(p, \tilde{g} + g' + \sum_{i=1}^k \varepsilon'_i \mu(F_i)\right)\right), \quad \forall (p, \tilde{g}) \in P^n \times (\mathbb{Z}_2)^{k+n}$$

Since  $\vartheta_{\hat{\lambda}_0} \circ \tilde{\zeta} = \Theta$ , so it is equivalent to write:

$$\left(\sum_{i=1}^k \varepsilon'_i e_i\right) \star \Theta(p, \tilde{g}) = \Theta\left(p, \tilde{g} + \sum_{i=1}^k \varepsilon'_i (e_i + \mu(F_i))\right). \quad (21)$$

Let  $H_\mu$  be the subgroup of  $(\mathbb{Z}_2)^{k+n}$  spanned by  $\{e_1 + \mu(F_1), \dots, e_k + \mu(F_k)\}$ . Since  $\mu$  takes value in  $(\mathbb{Z}_2)^n = \langle e_{k+1}, \dots, e_{k+n} \rangle$ , the rank of  $H_\mu$  is equal to  $k$ . Let  $\sigma : (\mathbb{Z}_2)^k \rightarrow H_\mu$  be a group isomorphism defined by

$$\sigma(e_i) = e_i + \mu(F_i), \quad i = 1, \dots, k. \quad (22)$$

Then according to (19) — (21), we have:

$$g' \star \Theta(p, \tilde{g}) = \sigma(g') \circledast \Theta(p, \tilde{g}), \quad \forall (p, \tilde{g}) \in P^n \times (\mathbb{Z}_2)^{k+n} \quad (23)$$

This implies that the free  $(\mathbb{Z}_2)^k$ -action on  $\mathcal{Z}_{P^n}$  defined by (18) is equivalent to the restriction of the canonical  $(\mathbb{Z}_2)^{k+n}$ -action on  $\mathcal{Z}_{P^n}$  to  $H_\mu$ . By combining the (19) and (23), we get:

$$\tilde{\Psi}(g' \cdot \theta_{\lambda_0}(x, g)) = \sigma(g') \circledast \tilde{\Psi}(\theta_{\lambda_0}(x, g)), \quad \forall \theta_{\lambda_0}(x, g) \in M(V^n, \lambda_0) \quad (24)$$

So we have proved the following proposition.

**Proposition 4.2.** *The natural free  $(\mathbb{Z}_2)^k$ -action on  $M(V^n, \lambda_0)$  is equivalent to the restriction of the canonical  $(\mathbb{Z}_2)^{k+n}$ -action on  $\mathcal{Z}_{P^n}$  to  $H_\mu$ .*

**Corollary 4.3.** *For any subgroup  $N \subset (\mathbb{Z}_2)^k$ , the natural  $(\mathbb{Z}_2)^k/N$ -action on  $M(V^n, \lambda_0)/N$  is equivalent to the canonical  $H_\mu/\sigma(N)$ -action on  $\mathcal{Z}_{P^n}/\sigma(N)$ .*

In addition, it is easy to see that the intersection of  $H_\mu$  with the isotropy subgroup of any orbit of  $\mathcal{Z}_{P^n}$  under the canonical  $(\mathbb{Z}_2)^{k+n}$ -action is trivial. So the canonical action of  $H_\mu$  on  $\mathcal{Z}_{P^n}$  is indeed free.

**Remark 4.4.** The equivalence  $\tilde{\Psi}$  identifies the orbit space  $M(V^n, \lambda_0)/(\mathbb{Z}_2)^k \cong Q^n$  with the partial quotient  $\mathcal{Z}_{P^n}/H_\mu$  (see (24)). Notice that if we choose another vertex  $v'_0$  of  $P^n$  and let  $F'_1, \dots, F'_k$  be the facets of  $P^n$  that are not incident to  $v'_0$ , we will get another subtorus  $H'_\mu$  of  $(\mathbb{Z}_2)^{k+n}$  with rank  $k$  by the above arguments so that  $Q^n \cong \mathcal{Z}_{P^n}/H'_\mu$  too. So the subtorus  $H \subset (\mathbb{Z}_2)^{k+n}$  that satisfies  $\mathcal{Z}_{P^n}/H \cong Q^n$  is not unique.

**Proof of Proposition 1.5:** Suppose  $P^n$  has  $k+n$  facets  $F_1, \dots, F_{k+n}$  and  $\pi_\mu : Q^n \rightarrow P^n$  is a small cover with a characteristic function  $\mu$  on  $P^n$ . Let  $V^n$  be a  $\mathbb{Z}_2$ -core of  $Q^n$  with panels  $\{P_1, \dots, P_k\}$  described above. By Theorem 2.2, for any principal  $(\mathbb{Z}_2)^m$ -bundle  $M^n$  over  $Q^n$ , there exists a  $\lambda \in \text{Col}_m(V^n)$  so that  $M^n$  is equivalent to  $M(V^n, \lambda)$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ . In addition,  $M^n$  is connected implies that  $L_\lambda = (\mathbb{Z}_2)^m$  (see Theorem 2.3), and so  $m \leq k$ . Without loss of generality, suppose  $\{\lambda(P_1), \dots, \lambda(P_m)\}$  is a linear basis of  $L_\lambda$ . In addition, we consider  $(\mathbb{Z}_2)^m$  as a direct summand of  $(\mathbb{Z}_2)^k$  and choose  $\omega_1, \dots, \omega_{k-m} \in (\mathbb{Z}_2)^k$  so that  $(\mathbb{Z}_2)^k = L_\lambda \oplus \langle \omega_1 \rangle \oplus \dots \oplus \langle \omega_{k-m} \rangle$ . Let  $\{e_1, \dots, e_k\}$  be a linear basis of  $(\mathbb{Z}_2)^k$  defined by the following:

$$e_i = \lambda(P_i), \quad 1 \leq i \leq m; \quad e_{m+j} = \omega_j, \quad 1 \leq j \leq k-m. \quad (25)$$

As the above discussion, let  $\lambda_0$  be a maximally independent  $(\mathbb{Z}_2)^k$ -coloring of  $V^n$  defined by  $\lambda_0(P_i) = e_i$  for  $1 \leq i \leq k$ . And we let:

$$N_\lambda := \langle e_{m+1}, \dots, e_k \rangle = \langle \omega_1 \rangle \oplus \dots \oplus \langle \omega_{k-m} \rangle \subset (\mathbb{Z}_2)^k.$$

Then we define an action  $\tilde{\eta}$  of  $N_\lambda$  on  $M(V^n, \lambda_0)$  by: for any  $1 \leq j \leq k-m$ ,

$$\tilde{\eta}(e_{m+j})(\theta_{\lambda_0}(x, g)) := (\lambda(P_{m+j}) + \omega_j) \cdot \theta_{\lambda_0}(x, g) \stackrel{(17)}{=} \theta_{\lambda_0}(x, g + \lambda(P_{m+j}) + \omega_j).$$

Obviously, the  $N_\lambda$ -action on  $M(V^n, \lambda_0)$  defined by  $\tilde{\eta}$  is free. And by a parallel argument as in the proof of Lemma 2.8, we can show that the orbit space of this  $N_\lambda$ -action on  $M(V^n, \lambda_0)$  is homeomorphic to  $M(V^n, \lambda)$ . Moreover, let:

$$N_\lambda^* := \langle \lambda(P_{m+1}) + \omega_1, \dots, \lambda(P_k) + \omega_{k-m} \rangle \subset (\mathbb{Z}_2)^k.$$

The rank of  $N_\lambda^*$  is  $k - m$ . Obviously, the  $N_\lambda$ -action on  $M(V^n, \lambda_0)$  defined by  $\tilde{\eta}$  is equivalent to the restriction of the natural  $(\mathbb{Z}_2)^k$ -action on  $M(V^n, \lambda_0)$  to  $N_\lambda^*$ . Then  $M(V^n, \lambda)$  is homeomorphic to  $M(V^n, \lambda_0)/N_\lambda^*$ . Moreover, we can check that the natural action of  $(\mathbb{Z}_2)^k/N_\lambda^*$  on  $M(V^n, \lambda_0)/N_\lambda^*$  is equivalent to the natural  $(\mathbb{Z}_2)^m$ -action on  $M(V^n, \lambda)$ . So  $M^n \cong M(V^n, \lambda)$  is equivalent to  $M(V^n, \lambda_0)/N_\lambda^*$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .

On the other hand, Corollary 4.3 says that the natural action of  $(\mathbb{Z}_2)^k/N_\lambda^*$  on  $M(V^n, \lambda_0)/N_\lambda^*$  is equivalent to the canonical action of  $H_\mu/\sigma(N_\lambda^*)$  on  $\mathcal{Z}_{P^n}/\sigma(N_\lambda^*)$ . Then combining all these equivalences, we have shown that  $M^n$  is equivalent to the partial quotient  $\mathcal{Z}_{P^n}/\sigma(N_\lambda^*)$  with the canonical  $H_\mu/\sigma(N_\lambda^*)$ -action as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ . By the definition of  $\{e_1, \dots, e_k\}$  in (25) and the definition of  $\sigma$  in (22),  $\sigma(e_{m+j}) = \sigma(\omega_j) = \omega_j + \mu(F_{m+j})$  for any  $1 \leq j \leq k - m$ . So  $\sigma(N_\lambda^*) \subset H_\mu \subset (\mathbb{Z}_2)^{k+n}$  is generated by the set:

$$\{\sigma(\lambda(P_{m+1})) + \omega_1 + \mu(F_{m+1}), \dots, \sigma(\lambda(P_k)) + \omega_{k-m} + \mu(F_k)\}.$$

Notice that the choice for each  $\omega_i$  is not unique, so the subtorus  $H'$  of  $(\mathbb{Z}_2)^{k+n}$  that satisfies  $\mathcal{Z}_{P^n}/H' \cong M^n$  is not unique either.  $\square$

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